Stability analysis of the one- and two-dimensional problem

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1. Governing equation

In the limit of zero Reynolds number the equation governing rigid spherical particle motion (position, x and velocity, v) in a unperturbed host flow, with velocity u, is the equation finalized by **?**:

$$\begin{aligned}\frac{dx}{dt} &= v \\ m_p\frac{dv}{dt} &= 6\pi a\mu(u - v + \frac{1}{6}a^2\Delta u) \\ &\quad +(m_p + m_f)g + m_f\frac{Du}{Dt} \\ &\quad +\frac{m_f}{2}\frac{d}{dt}(u - v + \frac{1}{10}a^2\Delta u) \\ &\quad +6\pi a^2\mu\int_0^t \frac{d}{d\tau}\left(u - v + \frac{1}{6}a^2\Delta u\right)\frac{d\tau}{\sqrt{\pi\nu(t-\tau)}}\end{aligned}$$

This equation can be simplified by the assumption of heavy particles the fact that gravity is negligible, resulting in the equation below:

$$\begin{aligned}\frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\frac{v - u}{\tau}\end{aligned}$$

2. One dimensional problem

The above equation will be applied to the motion of particles in a one dimensional sinusoidal flow field, moving with constant speed of c. The resulting system of equations is give below:

$$\begin{aligned}\frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\frac{v - a\sin 2\pi(x - ct)}{\tau}\end{aligned}$$

where c is the wave speed and a the amplitude. The above equation can be non-dimensionalised by using velocity scale of the background fluid, a. In order to simplify the equation a co-ordinate transformation is applied where x is substituted by $x' + ct$. Applying the above

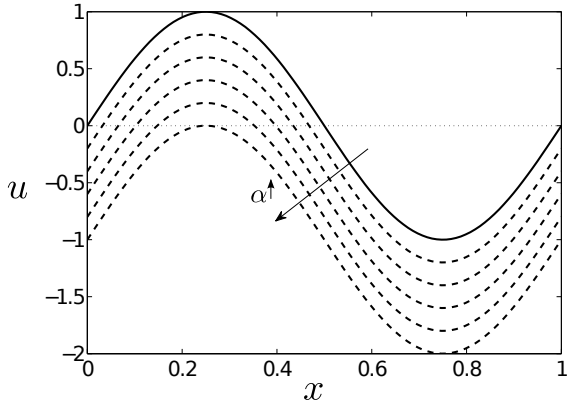


Figure 1: Velocity of the background fluid.

changes the final system of equations is given as;

$$\frac{dx'}{dt} = v'$$
$$\frac{dv'}{dt} = -\frac{v' + \alpha - \sin 2\pi x'}{St}$$

where St represents the Stokes number of the flow, and $\alpha = a/c$. The primes will be omitted in the following analysis for simplicity. The velocity profile of the background flow is shown in figure **??**. The solid line shows the velocity of the background fluid when $\alpha = 0$, and as the value of α is increased the profile shifts down until the point that $\alpha = 1$ and the velocity profile falls tangential to the axis of $u = 0$.

3. Instability analysis

The first step of performing the instability analysis is to determine the equilibria of the system of equations. As illustrated in figure **??** the value and the location of equilibria, where the velocity profile intersects the $u = 0$ axis, depends on the value of α. As the first step the special case of $\alpha = 0$ is considered, and finally the general case of $\alpha \neq 0$.

3.1. $\alpha = 0$

The equation of motion is linearized around the equilibria (x_0, v_0) by forming the Jacobian ,J as follows:

$$\begin{pmatrix} x \\ v \end{pmatrix} = J \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \tag{3.1}$$

The Jacobian for equilibrium point of $(0, 0)$ becomes

$$J = \begin{pmatrix} 0 & 1 \\ \frac{2\pi}{St} & -\frac{1}{St} \end{pmatrix} \tag{3.2}$$

The eigenvalues of the above matrix are, $\lambda_{1,2} = (-1 \pm \sqrt{1 + 8\pi St})/2St$. For all values of $St > 0$ there exist real eigenvalues of opposite signs that signifies a saddle point at this equilibrium. On the other hand for the other equilibrium point of $(\frac{1}{2}, 0)$ the linearized

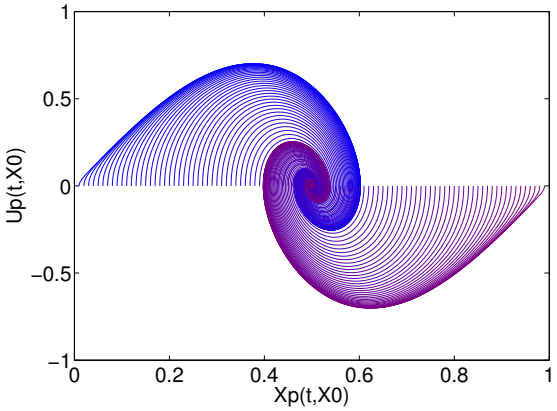
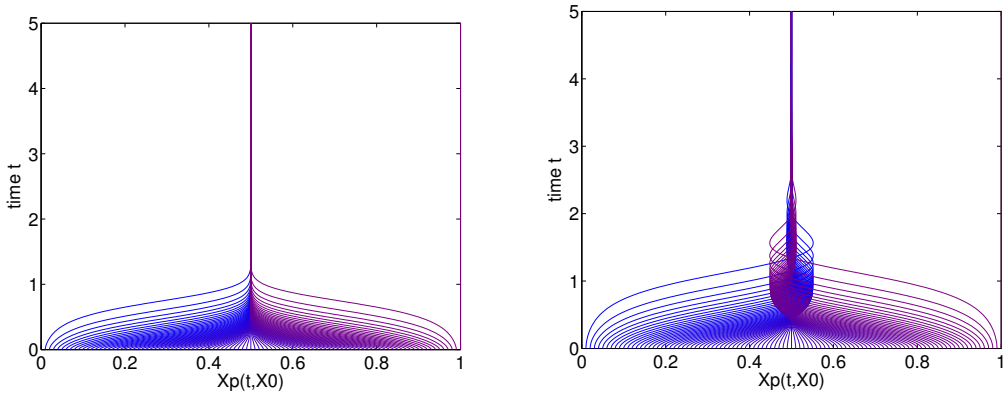


Figure 2: Evolution of particles in $x - v$ space for $\alpha = 0$



(a) $St = St_c/2$ (b) $St = 5St_c$

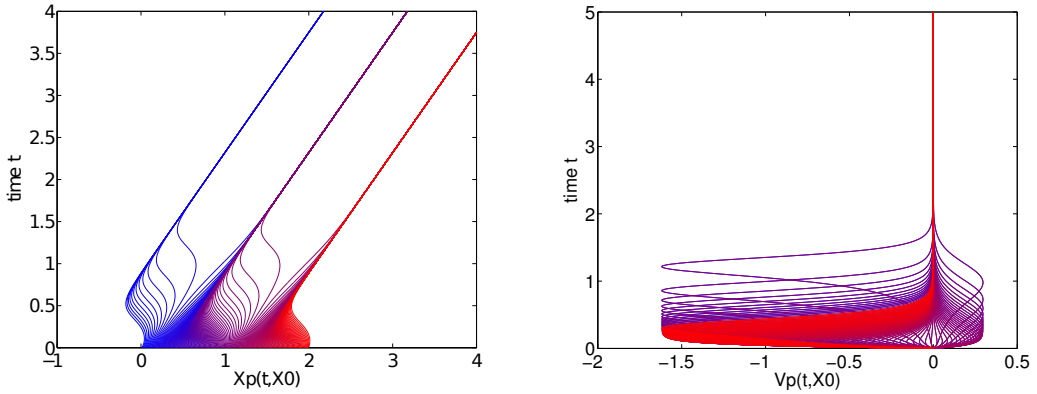
Figure 3: The evolution of particle position with time ($\alpha = 0$).

Jacobian can be written as;

$$J = \begin{pmatrix} 0 & 1 \\ -\frac{2\pi}{St} & -\frac{1}{St} \end{pmatrix} \tag{3.3}$$

The eigenvalues of this matrix are, $\lambda_{1,2} = (-1 \pm \sqrt{1 - 8\pi St})/2St$. For all values of the St the real part of the eigenvalues is negative which signifies a sink. These two points are shown in figure **??** where the trajectory of the particles are shown. The saddle and the sink are apparent in the figure.

However, for $St < 1/8\pi$ there is no imaginary part to the eigenvalue which defines monotonic convergence to the equilibrium point and for $St > 1/8\pi$ the eigenvalues contain imaginary part. This imaginary part shows non monotonic convergence to the sink, that suggest crossing. The trajectory of the particles for two values of St number below and above the critical Stokes number $St_c = 1/8\pi$ is shown in Figure **??**.



(a) Evolution of particle positions (b) Evolution of the particles in velocity space

Figure 4: The trajectories for $St < St_c$

3.2. $\alpha \neq 0$

As can be seen from figure **??** as long as $\alpha < 0$ there are still two equilibrium points and the stability around these points can be analyzed by linearizing the system of equation as done in the previous section. The two equilibria are $(\arcsin(\alpha)/2\pi, 0)$ and $(1/2 - \arcsin(\alpha)/2\pi, 0)$. For the first equilibria the Jacobian can be written as:

$$J = \begin{pmatrix} 0 & 1 \\ \frac{2\pi(\sqrt{1-\alpha^2})}{St} & -\frac{1}{St} \end{pmatrix} \tag{3.4}$$

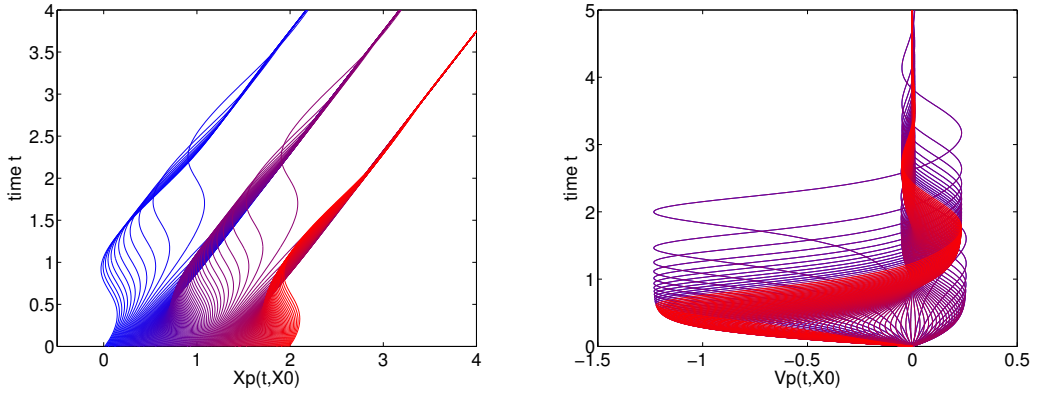
The eigenvalue of the above matrix are, $\lambda_{1,2} = (-1 \pm \sqrt{1 + 8\pi(\sqrt{1-\alpha^2})St})/2St$. Since $0 < \alpha < 1$ the two eigenvalues are real and of opposite signs. This suggests that this point is a saddle point. The Jacobian for the second equilibria is given as bellow.

$$J = \begin{pmatrix} 0 & 1 \\ -\frac{2\pi(\sqrt{1-\alpha^2})}{St} & -\frac{1}{St} \end{pmatrix} \tag{3.5}$$

The eigenvalue of the above matrix are, $\lambda_{1,2} = (-1 \pm \sqrt{1 - 8\pi(\sqrt{1-\alpha^2})St})/2St$. The same as done in the previous section as long as the Stokes number is smaller than the critical Stokes number, $St_c = 1/(8\pi\sqrt{1-\alpha^2})$ the eigenvalues will be real and negative. This suggests that the particles will be attracted to the sink monotonically. Figure **??** shows the trajectories of the particles in the two position and velocity space, x and v.

As, $St > St_c$ the eigenvalues become complex, but the real part remains negative, so the equilibria remains a sink, however due to the existence of the imaginary part to the eigenvalue the particles spiral toward this point. This suggests that the path would cross the zeroth location an oscillate towards the equilibrium point. Figure **??** shows the trajectories of the particles for this case in the position and velocity space.

In summary the evolution of the equilibria was calculated versus the variation of one of the parameters, α by using MatCont. Figure **??** shows this evolution. This figure shows the existence of two equilibria in each period of the background sinusoidal velocity, which agrees with the schematic of figure **??**. The solid line shows the unstable and the dashed



(a) Evolution of the particles in space (b) Evolution of the particles in velocity space

Figure 5: The trajectories for $St > St_c$

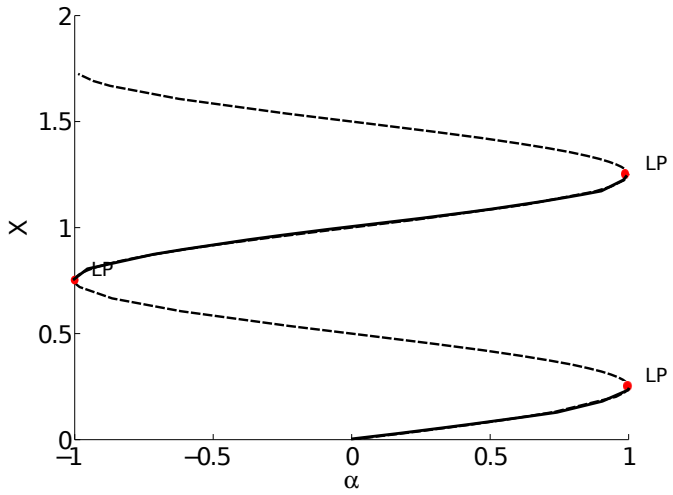
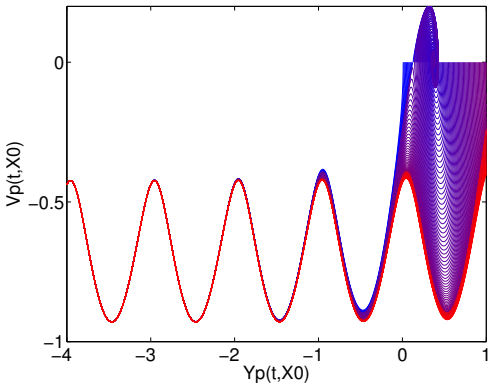


Figure 6: Evolution of the equilibria versus parameter α.

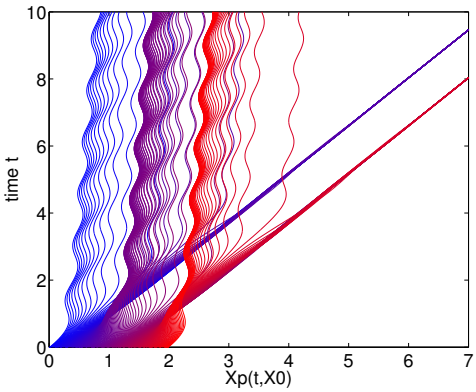
line the stable fixed points. The figure also illustrates the existence of limit points in the bifurcation diagram as parameter α reaches ± 1.

However, as the Stokes number is further increased form the initial critical value computed previously a different behavior is observed in the evolution of the particles. As can be seen in figure **??**(a), where the evolution of particles in the $x - v$ space is plotted, a set of particles are not attracted to the sink point and start drifting downstream. Figure **??**(b), (c) which show the evolution of position and velocity of the particles illustrate an oscillatory behavior for the position and velocity of these set of particles. This behavior is not explained through linearized equation and is due to the nonlinearity in the equation.

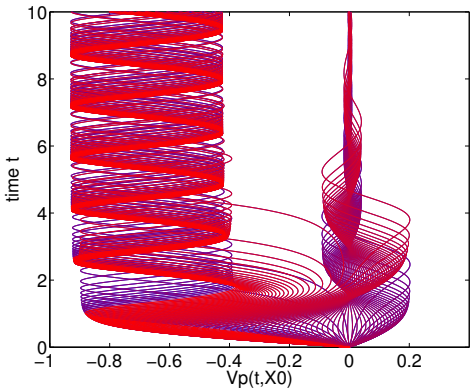
This behavior is better explained by looking at the phase space as the Stokes number



(a) Evolution of the particles in $x - v$ space



(b) Evolution of the particles in space



(c) Evolution of the particle velocity

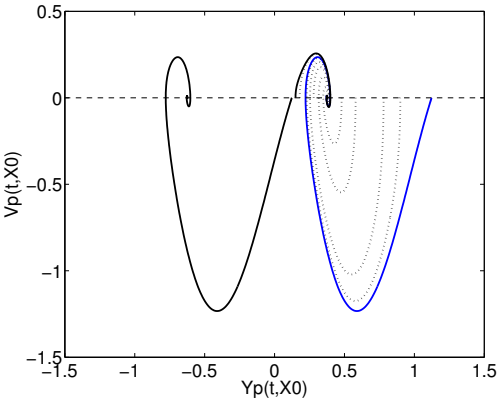
Figure 7: The trajectories for $St > St_{cc}$

passer through the second critical value, shown in Figure **??**. Figure **??**(a) shows the phase space where the Stokes number is higher than the first critical value but is still smaller than the second critical number. This figure shows that all trajectories are attracted to the stable fixed points of the system. However as the Stokes number reaches the second critical value a homoclinic orbit appears in the phase space. As the Stokes number is further increased the basin of attraction of the stable fixed point shrinks, and the particles starting out side of this basin of attraction travel towards a second attractor now present in the phase space which appears to have a limit cycle.

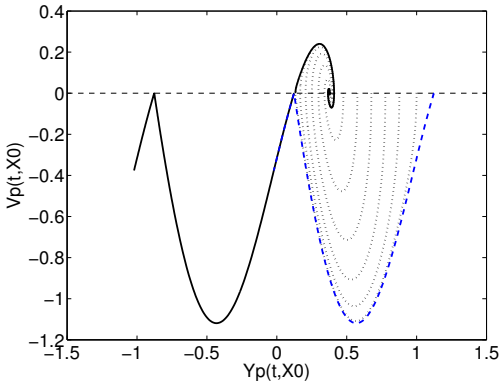
The evolution of the two critical Stokes number is plotted versus parameter α in Figure **??**. The plot shows different characteristics of the two Stokes number as one seems to decrease and the other increase with increasing α.

4. Taylor-Green vortices

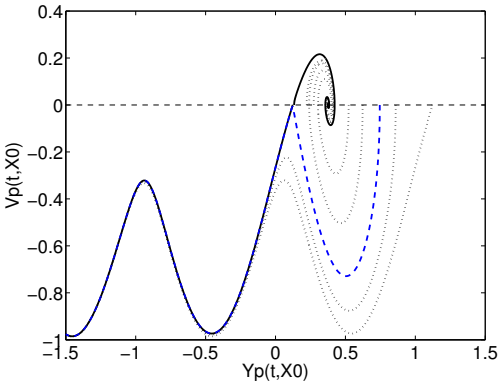
As the next step a two dimensional case of Taylor-Green vortices are examined and the results are compared to the one dimensional case. The evolution of particles in the



(a) $St < St_{cc}$



(b) $St = St_{cc}$



(c) $St > St_{cc}$

Figure 8: Evolution in the phase space.

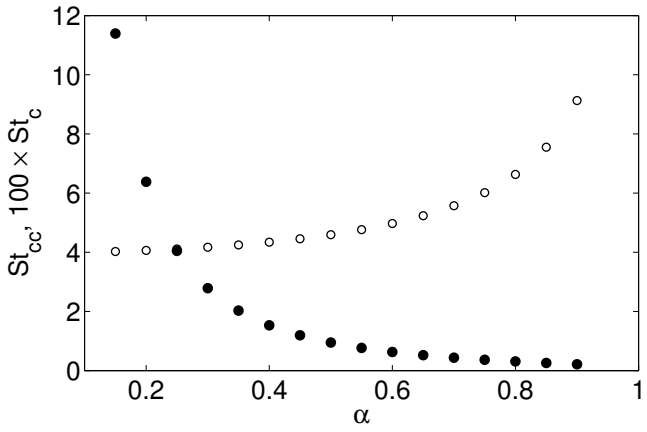


Figure 9: Evolution of the Stokes numbers, •, St_{cc}; ◦, St_c.

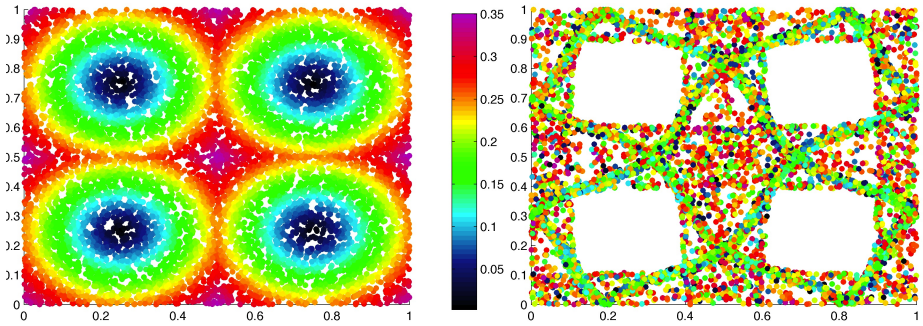


Figure 10: (left) particles in the Taylor-Green vortex initialized by the distance from the center. (right) the location of the particles at $t = 4s$.

space are shown in figure **??**, Stokes number is chosen to be higher than St_c. Due to the choice of the St number we expect crossing to appear at the sides of the lattice. This agrees well with what is shown in the figure.

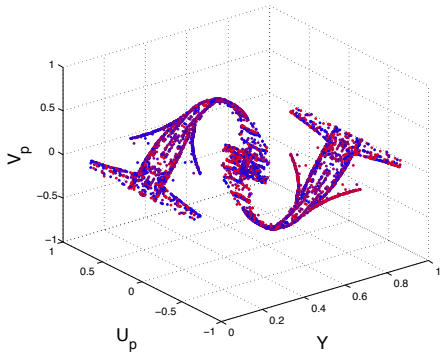
In order to study the phase diagram of the above system a Poincaré section at $x = 0.5$ is chosen and the location of particles in the phase space is plotted for P^{50}. As the initial condition the the phase space $\{u, v, y\} = \{[-1\ 1], [-1\ 1], [0\ 1]\}$ is populated with 8000 particles equally spaced in each direction. The Stokes number is chosen as $St = 5St_c$. Figure **??** shows four different views of the Poincaré section.

There seems to be two attractors in the phase space, one is the motion due to the attraction of particles to the sink which appears in the form of random motion at the location of the lattice and one which is more obvious in the phase-space of velocity elements due to the drift of particles with periodic motion. In order to differentiate the two separate behaviors more clearly the trajectories of particles with time is plotted in figure **??**(a). The periodic motion of the particles due to drift is shown in the plot by four beams stemming from the center of the figure. The chaotic motion in the velocity phase-space is shown in the center of the figure where particles are attracted to the vicinity of the latices. If we project everything in the space $\{x, y\} \in [0\ 1], [0\ 1]$ by taking advantage of the periodicity in both spatial directions, which is plotted in figure **??**(b), we see the two attractors superimposed on each other. This figure is the same as figure **??**(right), with particles colored in such a way that the two attractors can be easily identified.

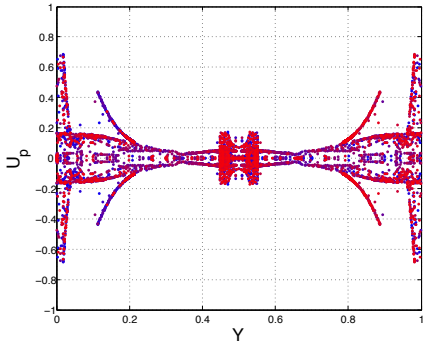
Figure **??** shows the dependance of the two attractors on the Stokes number in the velocity phase-space. This figure agrees with the Poincaré sections we produce taking larger Stokes numbers.

REFERENCES

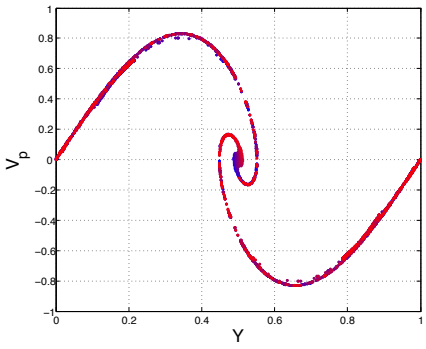
Maxey, M. R. & Riley, J. J. 1983 Equation of motion for a small rigid sphere in a nonuniform flow. *Phys. Fluids* **26**, 883.



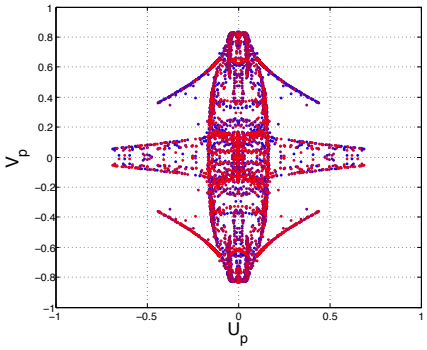
(a) Three-dimensional Poincaré section



(b) $y - u$ view of the Poincaré section

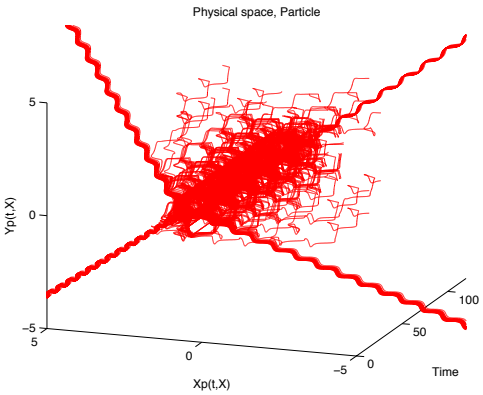


(c) $y - v$ view of the Poincaré section

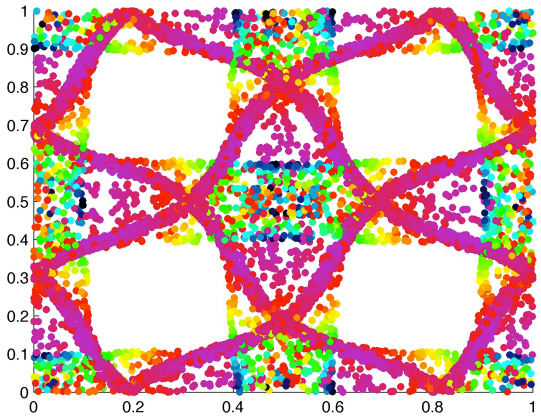


(d) $u - v$ view of the Poincaré section

Figure 11: Poincaré section for $St = 5St_c$.



(a) Particle trajectories



(b) Particle trajectories in periodic space

Figure 12: Evolution of the particle trajectories with time, $St = 5St_c$.

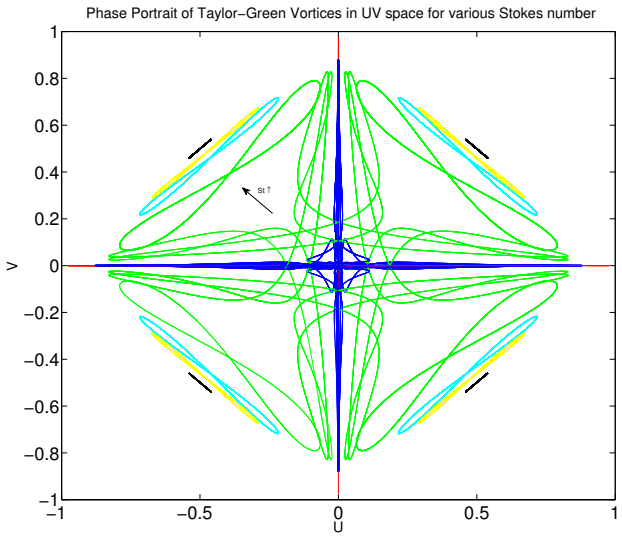


Figure 13: Evolution of the velocity phase-space with increasing Stokes number.